Post-Minkowskian Spinning Binary Dynamics in the Worldline Effective Field Theory Approach

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Work [arXiv:2102.10059] with Zhengwen Liu and Rafael A. Porto

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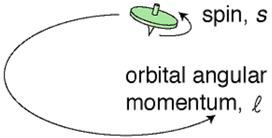
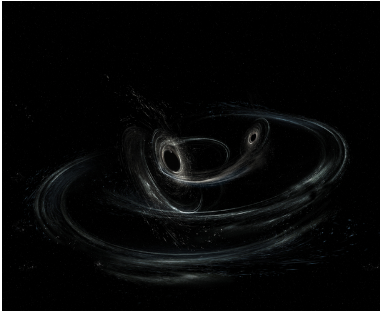
- Classical rotating angular momentum carried by the black holes. Total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$



- Spin can lead to significant corrections to the orbital motion of the compact binary, which results in the modulations on the gravitational-wave signal.



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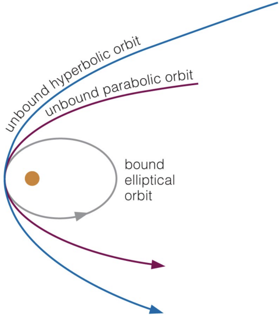
Post-Minkowskian Expansion

- A weak-field approximation $GM/rc^2 \ll 1$ in a background Minkowski spacetime
- No restriction on the relative velocity of the binaries (Contrast to Post-Newtonian expansions)
- Naturally applies to the weak-field scattering processes valid for all velocities



Newtonian two-body orbits.
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Conservative scattering of massive particles with spin and extend to bound orbits with aligned spins.

Worldline Effective Field Theory

Worldline action describing the relativistic point particles coupled with gravity

$$S_{\text{eff}}[x^\mu, g_{\mu\nu}] = S_{\text{EH}}[g] + S_{\text{pp}}[x, g],$$

with the Einstein-Hilbert action

$$S_{\text{EH}} = -2m_{Pl}^2 \int \mathrm{d}^4 x \sqrt{g} R(x),$$

and the point particle action

$$S_{\text{pp}} = -\sum_A \frac{m_A}{2} \int \mathrm{d}\tau_A g_{\mu\nu}\left(x_A\left(\tau_A\right)\right) v_A^\mu\left(\tau_A\right) v_A^\nu\left(\tau_A\right) + \ldots,$$

where the parametrized worldline $x_A^\mu(\tau_A)$ contains the information of the dynamics. Expanding the metric in the weak field approximation

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{\text{Pl}}}$$

Spinning Extended Objects

Anti-symmetric spin tensor $S^{\alpha\beta}$ describing the rotational degrees of freedom

- is constrained by the covariant spin supplementary condition (SSC): $S^{\alpha\beta} p_\beta = 0$.
- projected onto the locally-flat frame $S^{ab} \equiv S^{\mu\nu} e^a_\mu e^b_\nu$, with the co-rotating tetrad e^a_μ satisfying $g^{\mu\nu} e^a_\mu e^b_\nu = \eta^{ab}$.
- S^{ab} obeys $\{S^{ab}, S^{cd}\} = \eta^{ac} S^{bd} + \eta^{bd} S^{ac} - \eta^{ad} S^{bc} - \eta^{bc} S^{ad}$.
- Spin four-vector is defined as $S^\mu_A \equiv \dfrac{1}{2m_A} \epsilon^\mu{}_{\nu\alpha\beta} S^{\alpha\beta}_A p^\nu_A$.

The point-particle worldline action is extended to $S_{\mathrm{pp}} \equiv \int_{-\infty}^{+\infty} d\tau \mathcal{R}$, where the Routhian $\mathcal{R}$ is given by

$$\mathcal{R} = -\frac{1}{2}\Big(m g_{\mu\nu} v^\mu v^\nu + \omega^{ab}_\mu S_{ab} v^\mu + \frac{1}{m} R_{\beta\rho\mu\nu} e^\alpha_a e^\beta_b e^\mu_c e^\nu_d S^{ab} S^{cd} v^\rho v_\alpha - \frac{C_{ES^2}}{m} E_{\mu\nu} e^\mu_a e^\nu_b S^{ac} S^b_c + \cdots \Big)$$

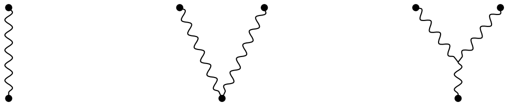
The last two terms account for the conservation of the SSC and finite-size effects to quadratic order in the spins.

Effective Routhian And The Impulses

Integrating out the potential fields

$$e^{iS_{\text{eff}}\left[x_A, S_A^{ab}\right]} = \int \mathcal{D}h_{\mu\nu} e^{iS_{\text{EH}}[h] + iS_{\text{GF}}[h] + i\int d\tau \mathcal{R}\left[x_A, S_A^{ab}, h\right]}$$

involving calculating the Feynman diagrams



(a) (b) (c)

As a result, $S_{\text{eff}} = \sum_n \int d\tau_1 \mathcal{R}_n \left[x_1(\tau_1), S_1(\tau_1); x_1(\tau_2), S_2(\tau_1)\right]$

The total change of linear momentum and spin tensor

$$\Delta p_A^\mu = -\eta^{\mu\nu} \sum_n \int_{-\infty}^{+\infty} \mathrm{d}\tau_A \frac{\partial \mathcal{R}_n}{\partial x_A^\nu},$$

$$\Delta S_A^{ab} = \sum_n \int_{-\infty}^{+\infty} d\tau_A \{S_A^{ab}, \mathcal{R}_n\}$$

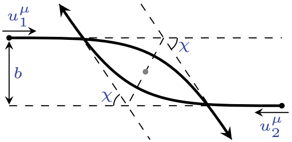
Scattering Momentum And Spin Impulses

The solutions to the EoM in powers of G,

$$x_A^\mu(\tau_A) = b_A^\mu + u_A^\mu \tau_A + \sum_n \delta^{(n)} x_A^\mu(\tau_A),$$

$$v_A^\nu(\tau_A) = u_A^\nu + \sum_n \delta^{(n)} v_A^\nu(\tau_A),$$

$$S_A^{ab}(\tau_A) = \mathcal{S}_A^{ab} + \sum_n \delta^{(n)} S_A^{ab}(\tau_A),$$



The initial values $\{u_A^\mu, \mathcal{S}_A^{ab}, b_A^\mu\}$ that are related to incoming velocity, the initial spin, and the impact parameter, $b \equiv b_1 - b_2$

are solved iteratively with the $\mathcal{O}(G^n)$ contribution to the worldline Routhian

LO: $\mathcal{R}_1\left[b_A, u_A, \mathcal{S}_A^{ab}\right]$

NLO: $\mathcal{R}_2\left[b_A, u_A, \mathcal{S}_A^{ab}\right] + \mathcal{R}_1\left[b_A + \delta^{(1)} x_A^\mu(\tau_A), u_A + \delta^{(1)} v_A^\mu(\tau_A), \mathcal{S}_A^{ab} + \delta^{(1)} S_A^{ab}(\tau_A)\right]$

NNLO: ...

Momentum And Spin Impulse Results

- We have computed the LO and NLO results for Δp_A^μ and ΔS_A^μ to quadratic order in the spins, which include $\mathcal{O}(S_A), \mathcal{O}(S_A S_B)$ and $\mathcal{O}(S_A^2)$ effects.
- As a non-trivial check, the results are consistent with
  - the preservation of the SSC, $S_\mu p^\mu = 0$
  - the on-shell condition, $p^2 = m^2$
  - the constancy of the magnitude of the spin, $S_\mu S^\mu$ and $S_{ab} S^{ab}$,

  up to 2PM order.
- The scattering angle follows from the total change of momentum in the CoM $2 \sin\left(\frac{\chi}{2}\right) = \frac{\sqrt{-\Delta p_1^2}}{p_\infty}$ with the momentum at infinity p_∞, for the case of spins aligned with the orbital angular momentum.

The NLO impulses at $\mathcal{O}(S_A^2)$ are

$$\Delta_{a^2}^{(2)} p_1^\mu = \frac{\nu G^2 M^3}{|b|^4} \Big[D_5 a_{1\alpha} a_{1\beta} \left(T^{\alpha\beta\mu} + 5\hat{b}^\alpha \hat{b}^\beta \hat{b}^\mu\right) + D_6 a_{1\alpha}(a_1 \cdot u_2)\left(\Pi^{\alpha\mu} + 4\hat{b}^\alpha \hat{b}^\mu\right)$$
$$+ a_{1\alpha} a_{1\beta}\left(D_7 u_1^\mu - D_8 u_2^\mu\right)\left(\Pi^{\alpha\beta} + 4\hat{b}^\alpha \hat{b}^\beta\right) + \hat{b}^\mu\left(D_9 a_1^2 + D_{10}(a_1 \cdot u_2)^2\right)$$
$$+ 2D_6 {a_1}^\mu (a_1 \cdot u_2) + (a_1 \cdot u_2)^2\left(D_{11} u_1^\mu + D_{12} u_2^\mu\right) - a_1^2\left(D_{13} u_1^\mu - D_{14} u_2^\mu\right)\Big]$$
$$- (1 \leftrightarrow 2)$$

$$\Delta_{a^2}^{(2)} s_1^\mu = \frac{\nu G^2 M^3}{|b|^2} \epsilon^\mu{}_{\nu\alpha\beta} \Big[D_{26} a_{1\rho} a_{1\sigma} u_1^\beta u_2^\nu \left(T^{\alpha\rho\sigma} + 4\hat{b}^\alpha \hat{b}^\rho \hat{b}^\sigma\right)$$
$$+ {a_1}^\nu a_{1\sigma}\left(\frac{2}{3} D_5 u_1^\beta + D_1 u_{2\perp}^\beta\right)\left(\Pi^{\alpha\sigma} + 3\hat{b}^\alpha \hat{b}^\sigma\right)$$
$$+ \left(\Pi^{\nu\sigma} + 2\hat{b}^\sigma \hat{b}^\nu\right)\Big(D_{28} {u_1}^\alpha {u_2}^\beta \left(a_1^2 \hat{b}_\sigma - a_{1\sigma}(\hat{b} \cdot a_1)\right)$$
$$+ 2\gamma D_{27} \hat{b}^\alpha a_{1\sigma}(a_1 \cdot u_2)\left({u_{2\perp}}^\beta + \left(\gamma^2 - 1\right){u_1}^\beta\right) - D_{27}(a_1 \cdot u_2) a_1^\alpha \hat{b}_\sigma ({u_1}^\beta - 2\gamma {u_2}^\beta)$$
$$+ D_{28}(a_1 \cdot u_2) a_{1\sigma} \hat{b}^\alpha {u_1}^\beta + D_{29} \hat{b}^\beta a_1^\alpha a_{1\sigma}\Big) + D_{28} \hat{b}^\alpha {u_1}^\nu {u_2}^\beta a_{1\rho} a_{1\sigma}\left(\Pi^{\sigma\rho} + 2\hat{b}^\rho \hat{b}^\sigma\right)$$
$$+ \hat{b}^\nu {u_1}^\alpha {u_2}^\beta\left(D_{30} a_1^2 + D_{31}(a_1 \cdot u_2)^2\right) + D_{32} a_1^\nu {u_1}^\alpha {u_2}^\beta (a_1 \cdot \hat{b})$$
$$+ a_1^\nu \hat{b}^\alpha (a_1 \cdot u_2)\left(D_{33} u_1^\beta + D_{34} u_2^\beta\right) - \frac{2}{3} D_{10} a_1^\alpha u_1^\beta u_2^\nu (a_1 \cdot u_2)\Big]$$

with auxiliary tensors and D_i coefficients defined in the paper.

Aligned-spin B2B Map And Radial Action

Boundary-to-Bound (B2B) map

A dictionary between gravitational observables for scattering processes measured at the boundary and adiabatic invariants for bound orbits , to all orders in the PM expansion
Gregor Kälin and Rafael A. Porto [arXiv:1910.03008, 1911.09130]

The radial action $i_r(\mathcal{E}, \ell, a)$ and the radial momentum $P_r(\mathcal{E}, \ell, a)$ for the bound systems

$$i_r(\mathcal{E}, \ell, a) = \frac{1}{2\pi G M \mu} \oint P_r(\mathcal{E}, \ell, a) dr$$

computed from the PM coefficients of the scattering angle χ. The correspondence between the periastron advanced $\Delta\Phi$, and scattering angle χ,

$$\frac{\Delta\Phi(J, \mathcal{E})}{2\pi} = \frac{\chi(J, \mathcal{E}) + \chi(-J, \mathcal{E})}{2\pi}, \quad \mathcal{E} < 0$$

where $\mathcal{E}$ is the reduced binding energy.

Aligned-spin B2B Example: Binding Energy

The binding energy for circular orbits can be computed through the orbital angular momentum $\ell_c(\mathcal{E}_c, a_\pm)$ solving the condition $i_r\left(\ell_c, \mathcal{E}_c, a_\pm\right) = 0$. Using the PN parameter

$$x \equiv (GM\Omega_c)^{2/3} = \left(\frac{d\ell_c}{d\mathcal{E}_c}\right)^{-2/3}$$

and some algebra, we find that

$$\begin{aligned}
\epsilon_c = x - \frac{x^2}{12}(\nu+9) + x^{5/2}\left(\frac{1}{3}(\delta\tilde{a}_- + 7\tilde{a}_+) + \frac{x}{18}\Big[(99-61\nu)\tilde{a}_+ - (\nu-45)\delta\,\tilde{a}_-\Big]\right) \\
+ \frac{1}{6}x^3\Big[-(C_{ES_+^2}+2)\tilde{a}_+^2 - (C_{ES_+^2}-2)\tilde{a}_-^2 - 2C_{ES_-^2}\,\tilde{a}_-\tilde{a}_+\Big] \\
+ \frac{5}{72}x^4\Big[\big(6(\nu-5)C_{ES_-^2} - 4(3C_{ES_+^2}+5)\delta\big)\tilde{a}_-\tilde{a}_+ + \big(32 - 6\delta C_{ES_-^2} + 10\nu + 3(\nu-5)C_{ES_+^2}\big)\tilde{a}_-^2 \\
+ \big(20 - 6\delta C_{ES_-^2} + 6\nu + 3(\nu-5)C_{ES_+^2}\big)\tilde{a}_+^2\Big] + \cdots,
\end{aligned}$$

to 3PN order and quadratic in spin and it agrees with the known value in the literature.

Conclusion And Future Outlooks

In this work, we have

- used the worldline EFT formalism to compute the NLO momentum and spin impulses with generic initial conditions and to quadratic order in the spins.
- exploited the scattering angle with aligned-spin configurations to construct the bound radial action via the B2B correspondence, which leads to the gravitational observables for elliptic-like orbits, including
  - the periastron advance to $\mathcal{O}(G^2)$ and all orders in velocity;
  - the linear and bilinear in spin contributions to the binding energy for circular orbits to 3PN order;
  - Center-of-mass momentum in a quasi-isotropic gauge to 2PM.

In the future plans, we aim to

- compute the impulses from spin effects to higher order in the PM expansion.
- find the hidden Lorentz covariance of the momentum and spin impulse results in a more compact form.
- generalize the B2B correspondence to the case of non-aligned spins to include the precession of the orbital plane.