Quantum Field Theory from the LHC to the Einstein Telescope

Part II. Feynman meets Einstein

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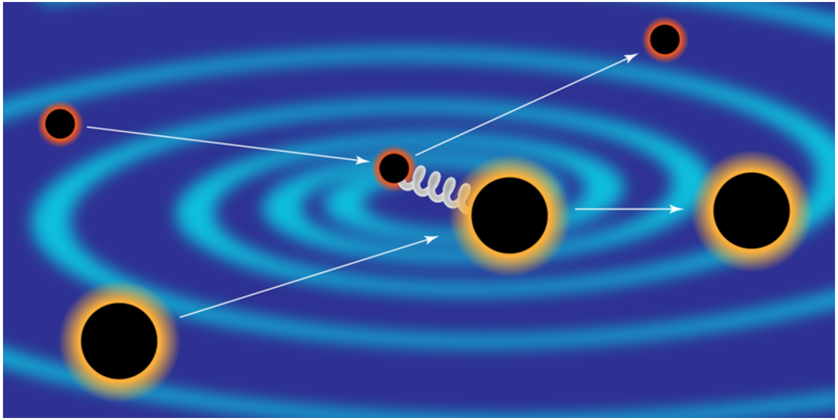
Work with G. Kälin & R. Porto [2007.04977, 2008.06047]

QU Day Nov 17, 2020

S-matrix meets Black Holes



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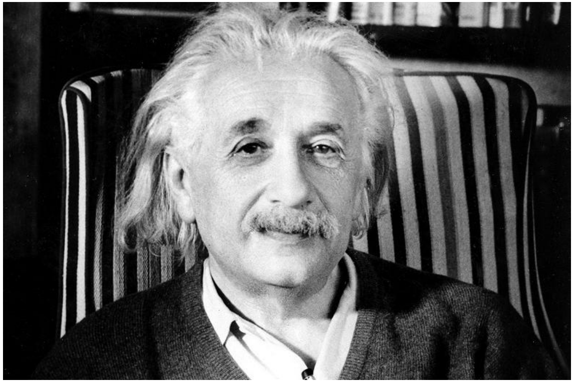
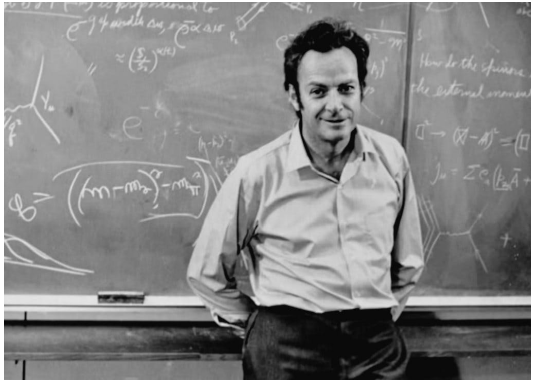
Black Holes: the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time. (S. Chandrasekhar)

Scattering amplitudes: the most perfect microscopic structures in the universe (L. Dixon)

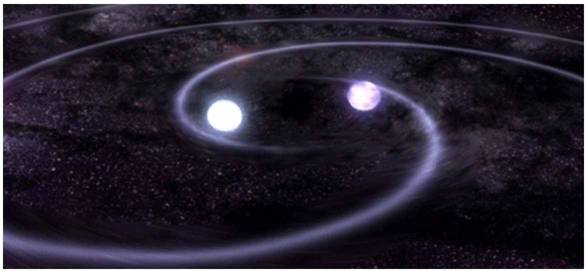
Ideas from amplitudes (QFT) are playing a crucial role in understanding (binary) black holes in the era of gravitational wave astronomy.

Feynman meets Einstien





I will talk about how to use advanced Feynman integration techniques to solve the classical two-body problem in Einstein's General Relativity.



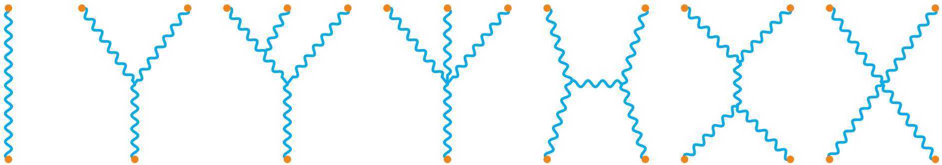
Post-Minkowskian EFT

- **Worldline EFT** [follow Gregor's talk]

$$\mathcal{L}_{\text{int}} = G\mathcal{L}_1 + G^2\mathcal{L}_2 + G^3\mathcal{L}_3 + \cdots$$

$$\Delta p_a^\mu = \int_{-\infty}^{\infty} d\tau_a \left(-\eta^{\mu\nu}\frac{\partial \mathcal{L}_{\text{int}}}{\partial x_a^\nu}\right)$$

- **Feynman diagrams**: 3rd order in Newton's constant G (aka 3PM)



- **Feynman integrals**

Kälin, ZL, Porto, 2007.04977

$$I^{(ab)}_{i_1 i_2; a_1\cdots a_5} = e^{2\gamma_E\epsilon}\int \frac{d^Dk_1\, d^Dk_2}{\pi^D}\, \frac{\delta(k_1\cdot u_a)\delta(k_2\cdot u_b)}{A_1^{i_1}A_2^{i_2}D_1^{a_1}D_2^{a_2}D_3^{a_3}D_4^{a_4}D_5^{a_5}}$$

with

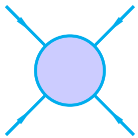
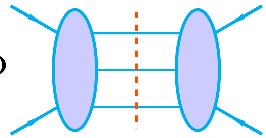
$$D_1 = k_1^2,\quad D_2 = k_2^2,\quad D_3 = (k_1+k_2-q)^2,\quad D_4 = (k_1-q)^2,\quad D_5 = (k_2-q)^2$$

$$A_1 = k_1\cdot u_{\not a} \pm i0,\quad A_2 = k_2\cdot u_{\not b} \pm i0 \qquad q\cdot u_a = 0,\quad u_a^2 = 1,\quad u_1\cdot u_2 = \gamma$$

The dependence of q^2 is fixed by their mass dimensions; they are single-scale integrals!

Reverse Unitarity

Optical theorem

Im  $\Longleftrightarrow \int d\Phi$ 

Reverse Unitarity: use the optical theorem backwards and express inclusive phase space integrals as unitarity cuts of loop integrals. [Anastasiou, Melnikov '02; Anastasiou, Dixon, Melnikov, Petriello '03]

LHC physics: it has been successfully used to do precision computations for Higgs production to N^{3}LO at LHC. Anastasiou, Duhr, Dulat, Herzog, Mistlberger, 1503.06056

GW physics: replace the delta-function by the cut-propagator Kälin, ZL, Porto, 2007.04977

$$\delta(k_i \cdot u_a) \to \frac{1}{2\pi i}\left(\frac{1}{k_i \cdot u_a - i0} - \frac{1}{k_i \cdot u_a + i0}\right)$$

Then standard loop-integral techniques can be applied straightforwardly!

IBP & Master Integrals

Integration-By-Parts (IBP):

$$\int \prod_i d^D k_i \, \frac{\partial}{\partial k_j^\mu} \left(\frac{v^\mu}{D_1^{a_1} D_2^{a_2} \cdots} \right) = 0$$

IBP gives a large homogeneous linear system of Feynman integrals.

Master Integrals: By solving the IBP identities, we can write many integrals as a linear combination of a small number of basis integrals $\{I_1, I_2, \ldots\}$:

$$I = \sum_i c_i \, I_i$$

Publicly available programs: `FIRE`, `LiteRed`, `Kira`, `Reduze`, `AIR`.

3PM master integrals

$$\left\{ I^{(12)}_{00;11011},\ I^{(12)}_{00;11100},\ I^{(12)}_{00;00211},\ I^{(12)}_{00;21100},\ I^{(12)}_{00;10110},\ I^{(12)}_{00;11111},\ I^{(12)}_{00;11211},\ I^{(12)}_{11;11100} \right\}$$

Precision predictions rely mainly on our ability to evaluate (master) integrals.

Multiple Polylogarithms

- Which numbers/functions appear in analytic results for Feynman integrals? Duhr 1411.7538
- A large class of functions are Goncharov's multiple polylogarithms (MPLs) math/0103059

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t), \;\; G(\,;z) = 1, \;\; G(\vec{0}_n; z) = \frac{1}{n!} \log^n(z)$$

- MPLs contain the well-known classical polylogarithms

$$\mathrm{Li}_n(z) = -G(\vec{0}_{n-1}, 1; z)$$

and harmonic polylogarithms (HPLs) Remiddi & Vermaseren, hep-ph/9905237

$$H_{\vec{a}}(z) = (-1)^p G(\vec{a}; z), \quad a_i \in \{0, \pm 1\}$$

- Transcendental weights $\mathcal{T}$: the number of iterated integrations in the definition

$$\mathcal{T}(1) = 0, \quad \mathcal{T}(\log x) = 1, \quad \mathcal{T}(\zeta_n) = \mathcal{T}(\mathrm{Li}_n 1) = n, \quad \mathcal{T}(\pi) = 1 \;\Longleftarrow\; \zeta_2 = \frac{\pi^2}{6}$$

Differential equations

- A powerful tool for analytic evaluation of master integrals:

$$\frac{d\vec{f}_0(x,\epsilon)}{dx} = \mathbb{M}_0(x,\epsilon)\vec{f}_0(x,\epsilon)$$

where $D = 4 - 2\epsilon$, $\vec{f}_0 = \{I_1, \ldots, I_N\}$.

- In 2013, Henn observed if all functions in a basis $\vec{f} = \{f_1, \ldots, f_N\}$ have uniform transcendental (UT) weights, the differential equations get dramatically simplified Henn 1304.1806

$$\frac{d\vec{f}(x,\epsilon)}{dx} = \epsilon\,\mathbb{M}(x)\,\vec{f}(x,\epsilon)$$

- Such basis is called the UT or canonical or pure basis. Then the master integrals can be solved in terms of Chen's iterated integrals (MPLs) in any order in ϵ. Henn 1412.2296
- Lee has proposed a complete algorithm to automate finding a canonical basis. Lee 1411.0911
Available programs: `epsilon`, `Fuchsia`, `Libra`, `CANONICA`, `INITIAL`.
- While these notions mainly originate in $\mathcal{N}=4$ SYM, they have been extensively applied in string theory, the Standard Model and recently GW physics. Arkani-Hamed et al. 1012.6032

Canonical basis

For our 3PM computations, the canonical basis takes

$$f_1 = \epsilon^4 \mathcal{I}_{11011}, \quad f_2 = \epsilon^4 \sqrt{\gamma^2-1}\, \mathcal{I}_{11100},$$

$$f_3 = \epsilon^3 \sqrt{\gamma^2-1}\, \mathcal{I}_{00211}, \qquad \mathcal{I}_{a_1\cdots a_5} \equiv I^{(12)}_{00;a_1\cdots a_5}$$

$$f_4 = -\frac{3\epsilon^4(\gamma^2-1)}{\gamma} \mathcal{I}_{11100} + \frac{\epsilon^3\,(\gamma^2-1)}{2\gamma} \mathcal{I}_{00211} + \frac{\epsilon^2\,(2\epsilon+1)}{2\gamma} \mathcal{I}_{21100}$$

$$f_5 = \frac{\epsilon^2\,(8\epsilon^2-6\epsilon+1)}{\sqrt{\gamma^2-1}} \mathcal{I}_{10110}, \quad f_6 = \epsilon^4 \sqrt{\gamma^2-1}\, \mathcal{I}_{11111},$$

$$f_7 = \frac{\epsilon^3\,\left(2(\gamma^2+1)\epsilon+1\right)}{2\gamma} \mathcal{I}_{11111} + \frac{\epsilon^3\,(\gamma^2-1)(\epsilon+1)}{2\gamma(2\epsilon+1)} \mathcal{I}_{11211} + \cdots$$

$$f_8 = \frac{1}{16}\,\epsilon^4\,(\gamma^2-1)\, I^{(12)}_{11;11111}$$

Kälin, ZL, Porto, 2007.04977 (a long paper in preparation)

Canonical differential equations

With the canonical basis, the differential equations take a nice form

$$d\vec{f} = \epsilon \Big(\mathbb{H}_0 \, d\log x + \mathbb{H}_+ \, d\log(1+x) + \mathbb{H}_- \, d\log(1-x) \Big) \, \vec{f}, \quad \gamma = \frac{x^2+1}{2x}$$

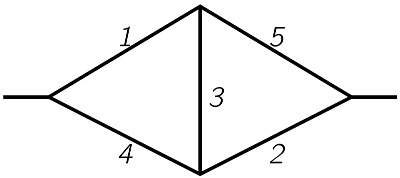
with

$$\mathbb{H}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 12 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 4 & 2 & -2 & 0 \\ 0 & 12 & 8 & 0 & 8 & 2 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{H}_\pm = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -4 & -2 & 0 & 0 \\ \pm 4 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Kälin, ZL, Porto, 2007.04977 Parra-Martinez, Ruf & Zeng, 2005.04236

Boundary conditions

- We take the static boundary condition: $\gamma = \frac{1}{\sqrt{1-\beta^2}} \to 1$ Kälin, ZL, Porto, 2007.04977

$\mathbf{K}_{a_1\cdots a_5} =$ 

$= \int \frac{d^{D-1}k_1 d^{D-1}k_2}{\pi^{D-1}} \frac{e^{2\gamma_E\epsilon}}{\mathbf{D}_1^{a_1}\mathbf{D}_2^{a_2}\mathbf{D}_3^{a_3}\mathbf{D}_4^{a_4}\mathbf{D}_5^{a_5}}$

- Only two master integrals are needed

$$\mathbf{K}_{11100} = (\mathbf{q}^2)^{-2\epsilon}\, e^{2\gamma_E\epsilon} \frac{\Gamma^3(1/2-\epsilon)\,\Gamma(2\epsilon)}{\Gamma(3/2-3\epsilon)}$$

$$\mathbf{K}_{11011} = (\mathbf{q}^2)^{-2\epsilon-1}\, e^{2\gamma_E\epsilon} \frac{\Gamma^4(1/2-\epsilon)\,\Gamma^2(\epsilon+1/2)}{\Gamma(1-2\epsilon)}$$

- Very interestingly, these integrals are building blocks in Post-Newtonian computations!
- PM integrals are naturally related to PN (boundary) integrals via differential equations!

Master integrals

$$f_1 = \epsilon^4 + \mathcal{O}(\epsilon^5), \qquad f_5 = 0$$

$$f_2 = \epsilon^3 \frac{\log(x)}{\pi^2} + \epsilon^4 \left(\frac{6H_{-1,0}(x)}{\pi^2} - \frac{6H_{1,0}(x)}{\pi^2} - \frac{3\log^2(x)}{\pi^2} - \frac{1}{2} \right) + \mathcal{O}(\epsilon^5)$$

$$f_3 = \epsilon^3 \frac{2\log(x)}{\pi^2} + \epsilon^4 \left(-\frac{4H_{-1,0}(x)}{\pi^2} + \frac{4H_{1,0}(x)}{\pi^2} + \frac{2\log^2(x)}{\pi^2} + \frac{1}{3} \right) + \mathcal{O}(\epsilon^5)$$

$$f_4 = -\frac{\epsilon^2}{\pi^2} + \epsilon^4 \left(\frac{8\log^2(x)}{\pi^2} + \frac{7}{6} \right) + \mathcal{O}(\epsilon^5)$$

$$f_6 = -\epsilon^3 \frac{2\log(x)}{\pi^2} + \epsilon^4 \left(-\frac{4H_{-1,0}(x)}{\pi^2} + \frac{4H_{1,0}(x)}{\pi^2} + \frac{2\log^2(x)}{\pi^2} + \frac{1}{3} \right) + \mathcal{O}(\epsilon^5)$$

$$f_7 = \epsilon^4 \left(\frac{12\log^2(x)}{\pi^2} + 2 \right) + \mathcal{O}(\epsilon^5)$$

$$f_8 = \epsilon^4 \left(\frac{\log^2(x)}{\pi^2} \right) + \text{b.c.} + \mathcal{O}(\epsilon^5)$$

Kälin, ZL, Porto, 2007.04977 (a long paper in preparation)   Parra-Martinez, Ruf & Zeng, 2005.04236

3PM results

- Impulse at 3PM order: Kälin, ZL, Porto, 2007.04977

$$\Delta p_1^\mu = \frac{G^3 b^\mu}{|b^2|^2}\left(\frac{16 m_1^2 m_2^2(4\gamma^4-12\gamma^2-3)}{(\gamma^2-1)}\,\textbf{Arcsinh}\sqrt{\frac{\gamma-1}{2}} - \frac{4m_1^2m_2^2\gamma(20\gamma^6-90\gamma^4+120\gamma^2-53)}{3(\gamma^2-1)^{5/2}} - \frac{2m_1m_2(m_1^2+m_2^2)(16\gamma^6-32\gamma^4+16\gamma^2-1)}{(\gamma^2-1)^{5/2}}\right)$$
$$+\frac{3\pi}{2}\frac{(2\gamma^2-1)(5\gamma^2-1)}{(\gamma^2-1)^2}\frac{G^3 m_1 m_2(m_1+m_2)}{|b^2|^{3/2}}\Big((m_1+\gamma m_2)u_2^\mu-(m_2+\gamma m_1)u_1^\mu\Big)$$

- We obtained the famous **arcsinh** function, first observed via a PN-type resummation.

[Bern, Cheung, Roiban, Shen, Solon, Zeng (BCRSSZ), 1901.04424 & 1908.01493]

$$\log x = -2\,\mathrm{Arcsinh}\sqrt{\frac{\gamma-1}{2}} \qquad \gamma = \frac{x^2+1}{2x}$$

- From the impulse we obtained the scattering angle. Perfect agreement with BCRSSZ!

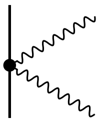
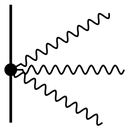
See Gregor's talk

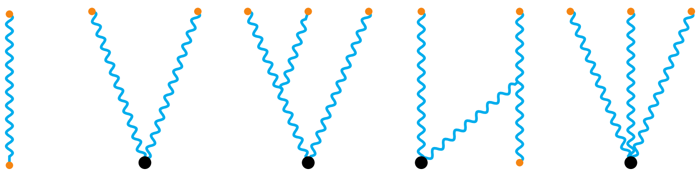
Finite-size effects

- As analogues of effective operators in the SM, we also studied tidal operators

$$-\frac{C}{4} H G^a_{\mu\nu} G^{a\mu\nu} \quad \Longleftrightarrow \quad \begin{array}{ll} \left(R_{\mu\alpha\nu\beta} v^\alpha v^\beta\right)^2, & \left(\nabla^\perp_{\{\mu} R_{\nu\alpha\rho\}\beta} v^\alpha v^\beta\right)^2 \\ \left(R^\star_{\mu\alpha\nu\beta} v^\alpha v^\beta\right)^2, & \left(\nabla^\perp_{\{\mu} R^\star_{\nu\alpha\rho\}\beta} v^\alpha v^\beta\right)^2 \end{array}$$

- These operators describe the tidal deformability of neutron stars. Kälin, ZL, Porto, 2008.06047

$$\mathcal{O}_{\text{tidal}} = \frac{1}{M_{\text{Pl}}^2}$$  $$+ \frac{1}{M_{\text{Pl}}^3}$$  $$+ \cdots$$



- We computed the quadrupolar and octupolar tidal corrections to NLO in PM expansion.
- Quadrupolar part is in perfect agreement with Cheung & Solon [2006.06665]
- We obtained the NLO octupolar corrections for the first time.
- All results are consistent with the test-body limit & the existing Post-Newtonian literature.

Conclusions

Many ideas and state-of-the-art techniques from QFT have already proven useful to improve theoretical predictions for Gravitational Wave observables:

- Effective Field Theory (see Gregor's talk)
- Scattering amplitudes & special functions
- Multi-loop techniques (IBP, differential eqs,...)

PM dynamics can be nicely bootstrapped from PN physics!

- The first complete classical derivation of conservative dynamics of binaries to 3PM order.
- Quadrupolar and octupolar tidal corrections to NLO.
- On-going: higher order, spin effects,...

Thanks for Your Attention !